GENERALISING FRACTIONAL BROWNIAN MOTION IN DISCRETE TIME: A SURPRISING SELF-SIMILARITY

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**Generalising in Discrete Time?**

**First we Restrict**

- Study the *fractional Gaussian Noise* (fGn)
- Equivalent to fBm in continuous time
- A simple time series in discrete time: \( X(k) = Z(t + k) - Z(t), \) \( k = 0, 1, 2, \ldots \)

**Then we Generalise**

- Motivated by idea of *second order self-similarity*
- Discrete fGn taken to be the only such process
- In fact *not*, if use natural renormalisation
SECOND ORDER STATIONARY TIME SERIES

Second order discrete time process \( \{X(t), t \in \mathbb{Z}\} \).
Mean \( \mu \) and variance \( \mathcal{V} > 0 \)
Autocovariance

\[
\gamma(k) := E[(X(t) - \mu)(X(t + k) - \mu)], \quad k \in \mathbb{Z}
\]

Autocorrelation

\[
\rho(k) := \frac{\gamma(k)}{\gamma(0)} = \frac{\gamma(k)}{\mathcal{V}}
\]
Equivalent Descriptors

Variance Time Function

\[ \omega(n) = \sum_{k=0}^{n-1} \sum_{i=-k}^{k} \gamma(i) = n\gamma(0) + 2 \sum_{i=1}^{n-1} i\gamma(n-i), \quad n = 1, 2, 3 \cdots, \]

Correlation Time Function

\[ \phi(n) = \frac{\omega(n)}{\omega(1)} = \frac{\omega(n)}{\mathcal{V}} \]

Note \( \omega(n) \) just variance of \( \sum_{t=1}^{n} X(t) \), and can invert: \( \gamma = \delta_n^2(\omega) \)

\[
\delta_n^2 \{f(i)\}(n) = \begin{cases} 
  f(1) & : n = 0 \\
  \frac{1}{2}(f(2) - 2f(1)) & : n = 1 \\
  \frac{1}{2}(f(n + 1) - 2f(n) + f(n - 1)) & : n > 1.
\end{cases}
\]

Note \( w(1) = \gamma(0) = \mathcal{V} \), so \( \phi(1) = \rho(0) = 1 \).
The *fractional noise* (\( \text{FN}_H \)) class of processes is defined by

\[
\rho_{\text{FN}}(k) = \delta_i^2 \{ i^{2H} \}(k)
\]

or equivalently

\[
\phi_{\text{FN}}(n) = n^{2H}
\]

where \( H \in [0, 1] \) is the Hurst parameter.

Note \( H = 0 \) is usually excluded!
For a fixed \( m \geq 1 \) the \textit{aggregation of level} \( m \) of the original process \( X \) is the process

\[
X^{(m)}(t) := \frac{1}{m} \sum_{j=m(t-1)+1}^{mt} X(j).
\]

Notation: \( \gamma^{(m)} \), \( \rho^{(m)} \), \( \omega^{(m)} \), \( \phi^{(m)} \) and \( \nu^{(m)} \) for the \( m \)-aggregation.
**Effect of Aggregation**

From $\gamma^{(m)}(k) = E[X^{(m)}(0)X^{(m)}(k)] - \mu^2$:

$$\gamma^{(m)}(k) = \frac{1}{m^2} \left[ m\gamma(km) + \sum_{i=1}^{m-1} i(\gamma((k-1)m + i) + \gamma((k+1)m - i)) \right].$$

And for the VTF

$$\nu^{(m)} = \gamma^{(m)}(0) = \frac{1}{m^2} \left( m\gamma(0) + 2 \sum_{i=1}^{m-1} i\gamma(m - i) \right) = \frac{\omega(m)}{m^2},$$

and so $\nu^{(mn)} = \frac{\omega(mn)}{(mn)^2} = \nu^{(m)}(n) = \frac{\omega^{(m)}(n)}{n^2}$, and so

$$\omega^{(m)}(n) = \frac{\omega(mn)}{m^2}.$$
Renormalisation

Aggregation rescales time, what about amplitude?

Natural choice is to normalise by the variance $\mathcal{V}^{(m)}$.

\[
\rho^{(m)}(k) = \frac{m \rho(km) + \sum_{i=1}^{m-1} i(\rho((k - 1)m + i) + \rho((k + 1)m - i))}{m \rho(0) + 2 \sum_{i=1}^{m-1} i \rho(n - i)}
\]

or

\[
\phi^{(m)}(n) = \frac{\phi(mn)}{\phi(m)}.
\]

Again, the correlation time formulation is simpler.
**Definition (Second-Order Self-Similarity)**

A process is second-order self-similar if $\rho^{(m)} = \rho$, or equivalently $\phi^{(m)} = \phi$, for all $m = 1, 2, 3, \cdots$.

What processes satisfy this?

Using the CTF, setting $\phi^{(m)} = \phi$, we obtain the fixed point equation

$$\phi(nm) = \phi(n)\phi(m)$$
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Using the CTF, setting $\phi^{(m)} = \phi$, we obtain the fixed point equation

$$\phi(nm) = \phi(n)\phi(m)$$
It is trivial to check that $\phi_{FN}(n) = n^{2H}$ is a solution.

Sinai showed (1976) that if power-law normalisation is used, then $FN_H$ is the only class of solutions.

Since $\mathcal{V}(m) = \mathcal{V}m^{2H-2}$ for $FN_H$, the two definitions (power-law or variance based normalisation) are equivalent in this important but special case.
GENERAL SOLUTION OF THE FIXED POINT EQUATION

\[ \phi(m) = \prod_{i=1}^{s} \phi(p_i)^{r_i}, \text{ for each } m = \mathbb{Z}^+, \]

where the \( p_i \) are the \( s \) distinct prime factors of \( m \), and \( r_i \) is the multiplicity of \( p_i \).
THREE SOLUTIONS

- FN_{0.6}
- Not Valid
- AP_{7,0.3}
TWO TRICKIER EXAMPLES

The Question

Background

Self Similarity

Almost Periodic

Classification

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The diagrams illustrate two examples of functions with different characteristics.

- **Left Diagram**: Shows a function $\rho(k)$ exhibiting self-similarity and randomness. The function oscillates around the zero line with varying amplitudes.

- **Right Diagram**: Displays a function $\rho(k)$ that is almost periodic with a repeating pattern. The function has a clear and regular oscillation with a consistent gap between the peaks.

These examples highlight the differences between self-similar and periodic functions, showcasing their distinct behaviors over the range of $k$ values shown.
A function is a *valid* autocovariance function if it is positive semi-definite. Recall that a function $f(k)$ defined on $k = 0, 1, 2, \ldots$ is said to be *positive semi-definite* if for any $n \in \mathbb{Z}, n > 0$ and for any real vector $a$ of length $n$

$$\sum_{1 \leq i, j \leq n} a_i f(|i - j|) a_j \geq 0.$$  

A 2nd order self-similar process must obey $\phi(nm) = \phi(n)\phi(m)$ and be valid.
The Almost Periodic Class

Definition (The Almost Periodic Family $\text{AP}_{q,c}$)

The two parameter family of fixed points defined by $\phi(1) = 1$, $\phi(p) = 1$ for all primes $p$ except $p = q$, where $\phi(q) = c$, $c \in (0, 1)$, will be called Almost Periodic, and denoted by $\text{AP}_{q,c}$. 
TWO EXAMPLES

The Question

Background

Self Similarity

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The diagrams illustrate two examples of functions and their logarithmic representations. The plots show the behavior of the functions $\phi(n)$ and $\rho(k)$ for different values of $n$ and $k$. The logarithmic plots provide insights into the scaling properties of these functions, which are characteristic of self-similar and almost periodic phenomena.
NEW SECOND-ORDER SELF-SIMILAR PROCESSES

DEFINITION (THE ALMOST PERIODIC FAMILY $\text{AP}_{q,c}$)

The two parameter family of fixed points defined by $\phi(1) = 1$, $\phi(p) = 1$ for all primes $p$ except $p = q$, where $\phi(q) = c$, $c \in (0, 1)$, will be called *Almost Periodic*, and denoted by $\text{AP}_{q,c}$.

THEOREM

Each member of the $\text{AP}_{q,c}$ family is a second-order self-similar process.
PROOF OF VALIDITY OF $\mathcal{AP}_{q,c}$ FAMILY

PROOF BY CONSTRUCTION

- Construct process with $\phi_m(n) = 0$ when $m|n$, else $\phi_m(n) = 1$.
- Define a new CTF based on an infinite sum:
  \[
  \phi_{q,c}(n) := \frac{1 - c}{c} \sum_{k=1}^{\infty} c^k \phi_{q^k}(n).
  \]
- This CTF is that of $\mathcal{AP}_{q,c}$!
- For partial sum, CTF is valid (process exists as sum of independent processes).
- Show CTF valid in limit using definition.
**Classification**

**Theorem (SS Process Classification)**

The set of valid fixed points is given by the union of the $FN_H$ and $AP_{q,c}$ families.
SKETCH OF PROOF

TWO CONSEQUENCES OF POSITIVE SEMI-DEFINITENESS

I  \[ \phi(m - n) \leq 2(\phi(m) + \phi(n)) \] for all \( m, n \in \mathbb{Z}^+ \), \( m - n \geq 1 \).

II  For any \( n \in \mathbb{Z}^+ \), \( |S(n)/\mathcal{V}| < 2\sqrt{\phi(n)} + C \),
     \( C \) a constant independent of \( n \).
     \[ S(n) = \sum_{k=-n}^{n} \gamma(k), \quad n \geq 0 \]

Powers of a prime \( p \) lie on unique power-law curve:

\[ f(x) = x^{\alpha_p}, \quad \text{where} \quad \alpha_p = \frac{\ln \phi(p)}{\ln p}. \]

Only if the \( \alpha_p \) are all equal is the fixed point of \( \text{FN}_H \) type.
SKETCH OF PROOF

CASE 1: \( \phi(p) > 1 \) FOR SOME PRIME \( p \)

- Implies \( \alpha := \sup_p \alpha_p > 0 \).
- If the \( \alpha_p \) equal, fixed point is of power-law type,
- If \( \alpha \in (0, 2] \rightarrow \text{valid} \) (just \( FN_H \) with \( H \in (0, 1] \)).

Assume then that the \( \alpha_p \) are not all equal.

CASE 1A: SUPRENUM \( \alpha \) NOT ATTAINED

- Implies \( \exists \) sequence \( p_i : \alpha_{p_i} > \max_{q<p_i} \alpha_q \)
- Can use (i) to show a contradiction \( \rightarrow \text{invalid} \)

CASE 1B: SUPRENUM \( \alpha \) ATTAINED

- Can find a \( q \) with \( \alpha = \alpha_q \), and other \( p \) with \( \alpha_p < \alpha \)
- Use (ii) and Euler’s theorem to show contradiction \( \rightarrow \text{invalid} \)
**Sketch of Proof**

**Case 2:** \( \phi(p) \leq 1 \) for all primes \( p \)

- Case of \( \phi = 1 \) is just \( FN_0 \rightarrow \text{valid} \).
- Case of only one prime \( q : \phi(q) < 1 \) included in \( AP_{q,c} \rightarrow \text{valid} \).
- Assume \( \exists \) different primes \( q_1 \) and \( q_2 \) for which \( \phi(q_i) < 1 \) (includes power-law fixed points with \( \alpha < 0 \) (\( H < 0 \))).
- Use fact that \( \phi(q_i^r) < \epsilon \) for \( r \) suff. large to show \( \phi(k) \rightarrow 0 \ \forall k \).
- Only satisfied by \( \phi = 0 \), impossible as \( V > 0 \rightarrow \text{invalid} \).
The Importance of $H = 0$

Theorem (The DoA of $FN_0$ is Large)

Let $X$ be any stationary process such that $\rho_\infty = \lim_{k \to \infty} \rho(k)$ exists and $\rho(1) < 1$. Then the differenced process $Y(i) = X(i + 1) - X(i)$ is in the domain of attraction of $FN_0$. 
A NEW CLASS OF SECOND ORDER SELF-SIMILAR PROCESSES

THE NATURE OF DISCRETE SECOND-ORDER SELF-SIMILARITY