A Characterisation Framework for Short and Long Memory Processes

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1. Confidence intervals for the mean (based on $\bar{X} = \sum_{i=1}^{n} X_i$)

2. Is the mean rate changing?
Background

Stationary second order processes:

\[ E[X] = \mu \quad \text{Var}[X] = \nu > 0 \quad \text{CoVar}[X_t, X_{t+k}] = \gamma(k) \]

Long-Range Dependent (LRD) processes, definitions:

1. \( \gamma(k) \sim c_\gamma k^{-\beta} \)
2. \( \gamma(k) \sim c_\gamma(k) k^{-\beta} \), \( c_\gamma(\cdot) \) slowly varying
3. \( \sum_{-\infty}^{\infty} \gamma(k) = \infty \), infinite covariance sum
4. more general?

Aggregation of level \( m \) (average over non-overlapping blocks of width \( m \)):

\[ X_i^{(m)} = \frac{1}{m} \sum_{m(i-1)+1}^{mi} X_k \]

Connection to mean estimation:

\[ \text{Var}[\bar{X}] = \nu^{(m)} = \text{Var}[X^{(m)}] \]
Objectives

Proving/Disproving/Clarifying “Folklore Results”:

- \( v^{(m)} \sim \frac{c_\gamma m^{2H-2}}{H(2H - 1)} \iff \gamma(k) \sim c_\gamma k^{2H-2} \), true? proven?
- fractional Gaussian noise (fGn) the only 2nd order Self-Similar (SS) process?
- all asymptotically 2nd order SS processes (ASS) are “fGn-like”?

Main New Results:

- “LRD” processes for which \( \gamma(k) \not\sim c_\gamma k^{2H} \).
- classification scheme for SS and ASS processes.
- proof for 
  most cases that SS processes must be fGn, but
- examples of SS “covariance functions” which are 
  not fGn-like.
Definitions

Begin with correlation function: \( \rho(k) = \gamma(k)/v, \quad (\gamma(k) \in [-1, 1]) \)

Integrate twice for two other quantities with physical meaning:

\[
S_2(i) = \sum_{k=-i}^{i} \gamma(k)
\]

\[
w(m) \equiv m^2 v^{(m)} = \sum_{i=0}^{m-1} S_2(i) = E[(X_1 + \ldots + X_m)^2] = \sum \text{matrix terms} \geq 0
\]

\(\gamma(k), \ S_2(i), \ w(m)\) are all equivalent!

\(\gamma(k) \rightarrow \gamma \quad \text{exists?} \)
\(S_2(i) \rightarrow S_2 \quad \text{exists?} \)
\(w(m) \rightarrow w \quad \text{exists?} \)

Typically all are defined, but have counterexamples for each.
The Fractional Gaussian Noise Fixed Point Family

Fractional Gaussian Noise, parameterised by $H$:

$$\rho_{fGn}(k) = \frac{1}{2} \left\{ |k - 1|^{2H} - 2|k|^{2H} + |k + 1|^{2H} \right\}$$

<table>
<thead>
<tr>
<th>Class</th>
<th>$H$</th>
<th>$S_2$</th>
<th>$w(\infty)$</th>
<th>$\rho^*(k), k \geq 3$</th>
<th>$\rho^*(k)$</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>1</td>
<td>$\rightarrow$</td>
<td>Non ergodic</td>
</tr>
<tr>
<td>A</td>
<td>$(0.5, 1)$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$(0, 1)$</td>
<td>$\downarrow$</td>
<td>LRD</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>$\nu^*$</td>
<td>$\infty$</td>
<td>0</td>
<td>$\rightarrow$</td>
<td>White noise</td>
</tr>
<tr>
<td>C</td>
<td>$(0, 0.5)$</td>
<td>0</td>
<td>$\infty$</td>
<td>$(-0.5, 0)$</td>
<td>$\uparrow$</td>
<td>anticorrelated</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>$\nu^*$</td>
<td>0</td>
<td>$\rightarrow$</td>
<td>MA(1) process</td>
</tr>
<tr>
<td>E</td>
<td>$&lt; 0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\downarrow$</td>
<td>not valid as a process</td>
</tr>
</tbody>
</table>
Aggregation, Fixed Points and Self-Similarity

Aggregation: \( X_i \leftrightarrow X_i^{(m)} \), \( (w(m) = m^2v^{(m)}) \),

\( \gamma() \leftrightarrow \gamma^{(m)}() \)  \( \rho() \leftrightarrow \rho^{(m)}() \)  \( w() \leftrightarrow w^{(m)}() \)

Definition \( \rho() \) is a **fixed point** or second order Self-Similar (SS) iff \( \rho^{(m)}() \equiv \rho(), \forall m \in \mathbb{Z}^+ \).

Theorems:

- \( w(mn)w(1) = w(m)w(n) \iff \rho^{(m)} \equiv \rho \)
- If \( \lim_{m \to \infty} \rho^{(m)}() \equiv \rho^*() \) then \( \rho^* \) is a fixed point.
  - say \( \rho \) is in the **Domain of Attraction (DoA)** of \( \rho^* \), call it ASS.
- \( \lim_{m \to \infty} w(mn)/w(m) = \phi(n) \) exists \( \iff \rho() \) is ASS.
  \( \phi(n) = n^{2H}, H \in [0,1] \iff \) limit is fGn-like.

Earlier Definitions of Fixed Points (SS):

- \( \rho^* = \rho_{fGn} \)
- \( \gamma^{(m)}(k) = \gamma(k)m^{-(1-H)} \) for all \( m \).
Taxonomy of Fixed Points: $w^*(mn)w^*(1) = w^*(m)w^*(n)$

Valid fixed points (+ve definite $\gamma(k)$)

Three kinds of fixed points:
- fGn-like
- strange, non-fGn-like. Do they exist?

Non-Valid fixed points (not +ve definite)
Non “LRD” Asymptotically SS Processes for $H \in [0, 1]$

Here $\rho(k) \not\propto c \rho(k)^{(2H - 2)}$, but $w^{(m)}$ regularly varying, so process is ASS to a fGn fixed point.
A New Kind of Self-Similar Process?

can 55 million people be wrong?...